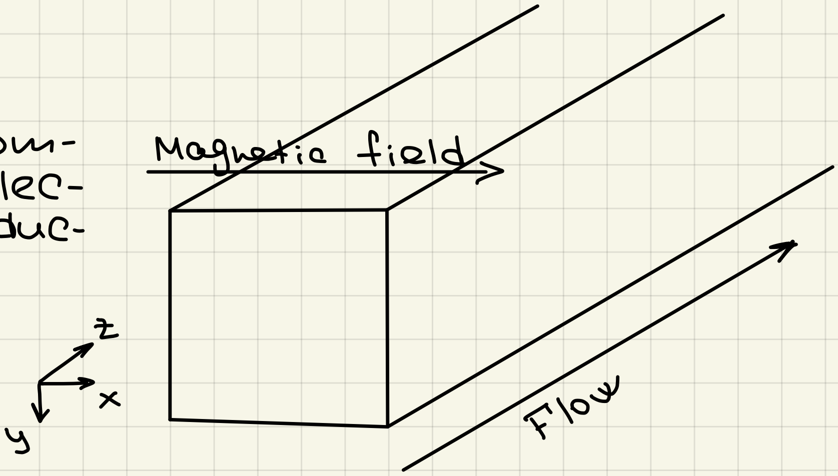
Magnetohydrodynamics
(MHD)

viscous incompressible electrically conducting fluid



I. Derive the MHD eq-s for this flow

non-dim.

$$\begin{cases} \bar{x}_{dim} \to L\bar{x}, \\ \bar{u}_{dim} \to U\bar{u}, \\ b_{dim} \to B_0 \bar{b} \end{cases} \qquad \begin{cases} \bar{u} = u(x,y)\,\bar{e}_z \\ p = p(z), \quad p'(z) = \text{const} \\ \bar{b} = \underbrace{\bar{e}_x}_{\text{applied}} + \underbrace{b(x,y)}_{\text{induced magn. field}}\,\bar{e}_z \end{cases}$$

$$\bar{\nabla}\cdot\bar{u} = 0,$$

$$\partial_t \bar{u} + \bar{u}\cdot\bar{\nabla}\bar{u} = -\bar{\nabla}p + \underbrace{\frac{1}{Re}}_{\sim \frac{U}{\nu}}\bar{\nabla}^2\bar{u} + \overbrace{\frac{Ha^2}{Re}\,\bar{j}\times\bar{b}}^{\text{the Lorentz force}},$$

$$\sim B_0^2$$

$$\partial_t \bar{b} + \bar{u}\cdot\bar{\nabla}\bar{b} = \bar{b}\cdot\bar{\nabla}\bar{u} + \frac{1}{Re_m}\bar{\nabla}^2\bar{b}, \qquad Re_m \sim U\sigma$$

$$\bar{\nabla}\cdot\bar{b} = 0$$

1) $$-\frac{\partial p}{\partial z} + \frac{1}{Re}\bar{\nabla}^2 u + \frac{Ha^2}{Re}(\bar{j}\times\bar{b})_z = 0,$$

$$(\bar{j}\times\bar{b})_z = \cancel{j_x b_y} - j_y b_x = -j_y$$

$$\bar{j} = \frac{1}{Re_m}\bar{\nabla}\times\bar{b} \quad \text{Ampère's eq-n}$$

$$j_y = -Re_m^{-1}\frac{\partial b}{\partial x}$$

$$\Rightarrow \bar{\nabla}^2 u + Ha^2 Re_m^{-1}\frac{\partial b}{\partial x} = p' Re = K$$

2) $\left(\frac{\partial}{\partial x} + b\frac{\partial}{\partial z}\right)u + Re_m^{-1}\nabla^2 b = Re_m^{-1}\nabla^2 b + \frac{\partial u}{\partial x} = 0$

$$\begin{cases} \nabla^2 u + Ha^2 Re_m^{-1}\dfrac{\partial b^{old}}{\partial x} = K, \\ Re_m^{-1}\nabla^2 b^{old} + \dfrac{\partial u}{\partial x} = 0. \end{cases}$$

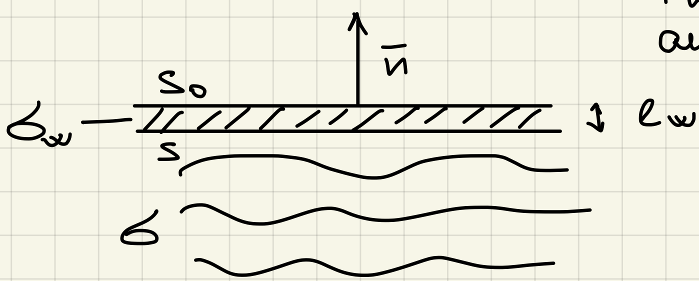
$$b = Re_m^{-1} b^{old}$$

$$\begin{cases} \nabla^2 u + Ha^2 \dfrac{\partial b}{\partial x} = K, \\ \nabla^2 b + \dfrac{\partial u}{\partial x} = 0. \end{cases} \quad \text{——} \quad \{u, b\}$$

BC: $u = 0$, b?

II. $\sigma_{w,1}$, $\sigma_{w,2}$, $\sigma_{w,3}$, $\sigma_{w,4}$ → given (electr. conductivity of each wall)

Assume $\underbrace{l_w}_{\text{thickness of the wall}} \ll \underbrace{L}_{\text{width of the duct}}$



In the general case we solve Maxw. eq-s in the fluid and in the wall and "connect" them → here, due to $l_w \ll L$ assump., we replace sol. in the wall with the BC.

Step 1. Use Maxwell's BC for the tangential component of $\bar{E}$ at the interface to derive the BC for the electric current.

$$(\bar{E} - \bar{E}_w)\cdot\bar{\tau} = 0$$

contin. of E_τ

$$\bar{j} = \sigma(\bar{E} + \bar{u}\times\bar{b})$$

$$\left(\frac{1}{\sigma}\bar{j} - \frac{1}{\sigma_w}\bar{j}_w\right)\cdot\bar{\tau} = 0,$$

$$(\bar{j}\cdot\bar{\tau})\bar{\tau} = -\bar{n}\times(\bar{n}\times\bar{j}) = \bar{n}\times(\bar{j}\times\bar{n})$$

$$\Rightarrow \left(\frac{1}{\sigma}\bar{j} - \frac{1}{\sigma_w}\bar{j}_w\right)\times\bar{n} = 0.$$

Step 2. Use Ampère's law to derive the BC for the induced magnetic field.

$$\bar{\nabla}\times\bar{b} = \mu_0 \bar{j}$$

$$\bar{j}\times\bar{n} = \frac{1}{\mu_0}(\bar{\nabla}\times\bar{b})\times\bar{n} = \frac{1}{\mu_0}\left[\bar{\nabla}\times(b\bar{e}_z)\right]\times\bar{n}$$

$$= \frac{1}{\mu_0}\left[\cancel{b\,\bar{\nabla}\times\bar{e}_z} + (\bar{\nabla}b)\times\bar{e}_z\right]\times\bar{n}$$

$$= \frac{1}{\mu_0}\bar{e}_z\underbrace{\left[\bar{n}\cdot(\bar{\nabla}b)\right]}_{\frac{\partial b}{\partial n}} - \cancel{\frac{1}{\mu_0}(\bar{\nabla}b)(\bar{e}_z\cdot\bar{n})}$$

$$\Rightarrow \qquad \frac{1}{\sigma}\frac{\partial b}{\partial n} - \frac{1}{\sigma_w}\frac{\partial b_w}{\partial n} = 0$$

Step 3. How can we utilize the assumption of a thin wall, $l_w \ll 1$, to express the BC for $\bar{b}$ in terms of flow variables only?

$$b|_s = b|_{s_0} + l_w\frac{\partial b_w}{\partial n} + \frac{l_w^2}{2}\frac{\partial^2 b_w}{\partial n^2} + \ldots$$

$$l_w \ll 1 \Rightarrow \quad b|_s = \underbrace{\cancel{b|_{s_0}}}_{\text{insul. medium outside the duct}} + l_w\frac{\partial b_w}{\partial n}$$

$$\frac{1}{l_w}b = \frac{\partial b_w}{\partial n}$$

$$\Rightarrow \boxed{\frac{1}{\sigma}\frac{\partial b}{\partial n} - \frac{1}{l_w\sigma_w}b = 0}$$

thin-wall BC

Non-dim.: $\bar{b}\to B_0\bar{b}$, $\bar{x}\to L\bar{x}$

$$\Rightarrow \frac{B_0}{L\sigma}\frac{\partial b}{\partial n} - \frac{B_0}{l_w\sigma_w}b = 0$$

$$\Rightarrow \frac{\partial b}{\partial n} - \frac{L\sigma}{l_w\sigma_w}b = 0$$

$\frac{1}{C}$ conductance ratio

Limit case: $\sigma_w \to 0$ (perf. insul. materials)

$\Rightarrow c = 0 \Rightarrow b = 0$

$\sigma_w \to \infty$ (perf. cond. materials)

$$\Rightarrow \frac{\partial b}{\partial n} = 0$$

The magnetic Reynolds number is a dimensionless parameter that estimates the relative contributions of advection and diffusion of a magnetic field in a conducting medium.

$$Re_m \sim \frac{\text{advection}}{\text{diffusion}} \sim \frac{|\bar{\nabla}\times(\bar{u}\times\bar{B})|}{|\eta\bar{\nabla}^2\bar{B}|} \sim \frac{U\cancel{B}}{\cancel{L}}\cdot\frac{L^{\cancel{2}}}{\eta\cancel{B}} = \frac{UL}{\eta}$$

$$\frac{\partial\bar{B}}{\partial t} = \underbrace{\bar{\nabla}\times(\bar{u}\times\bar{B})}_{\text{advection}} + \underbrace{\eta\,\bar{\nabla}^2\bar{B}}_{\text{diffusion}}$$

$$\Rightarrow Re_m = \frac{UL}{\eta}$$

For most industrial liquid metals Re_m is small:

$\eta \sim 1\ m^2/s$, $L \sim 0.1\ m$, $U \sim 0.01\ m/s$

$$\Rightarrow Re_m \sim 10^{-3}$$

For many liquid metals we commonly apply a so-called quasi-static approximation:

$$Re_m \ll 1.$$

$$\frac{\partial\bar{u}}{\partial t} + (\bar{u}\cdot\bar{\nabla})\bar{u} = -\frac{1}{\rho}\bar{\nabla}p + \nu\bar{\nabla}^2\bar{u} + \frac{1}{\rho}\bar{j}\times\bar{B},$$

$$\frac{\partial\bar{B}}{\partial t} = \bar{\nabla}\times(\bar{u}\times\bar{B}) + \eta\,\bar{\nabla}^2\bar{B},$$

$$\bar{\nabla}\cdot\bar{u} = 0,$$

$$\bar{\nabla}\cdot\bar{b} = 0.$$

Consider the motion of an incompressible liquid metal in a stationary uniform magnetic field. What are the quasi-static MHD equations?

$$\bar B = \bar B_0 + \bar b$$

$$\frac{\partial \bar B}{\partial t} = \bar\nabla\times(\bar u\times\bar B) + \eta\,\bar\nabla^2\bar B$$

$$\Rightarrow \frac{\partial \bar b}{\partial t} = \underbrace{\bar\nabla\times(\bar u\times\bar B_0) + \bar\nabla\times(\bar u\times\bar b)} + \eta\,\bar\nabla^2\bar b$$

1. $\bar\nabla\times(\bar A\times\bar C) = \bar A(\bar\nabla\cdot\bar C) - \bar C(\bar\nabla\cdot\bar A) + (\bar C\cdot\bar\nabla)\bar A - (\bar A\cdot\bar\nabla)\bar C$

$$\Rightarrow \underbrace{(\bar B_0\cdot\bar\nabla)\bar u}_{\sim\ \frac{B_0 U}{L}} - \cancel{(\bar u\cdot\bar\nabla)\bar B_0} + (\bar b\ \bar\nabla)\bar u - (\bar u\cdot\bar\nabla)\bar b$$

$$\frac{\eta|\bar\nabla^2\bar b|}{|(\bar u\cdot\bar\nabla)\bar b|} \sim \frac{\eta|\bar\nabla^2\bar b|}{|(\bar b\cdot\bar\nabla)\bar u|} \sim \eta\frac{\cancel{|\bar b|}}{L^{\cancel{2}}}\cdot\frac{\cancel{L}}{\cancel{|\bar b|}U} = \frac{1}{Re_m} \gg 1$$

$$\Rightarrow \frac{\partial \bar b}{\partial t} = (\bar B_0\cdot\bar\nabla)\bar u + \eta\,\bar\nabla^2\bar b$$ diffusion eq-n for $\bar b$ with ext. force

$$\frac{\partial \bar b}{\partial t} = \underbrace{\eta}_{\gg 1}\bar\nabla^2\bar b + \bar f(\bar x, t)$$

After some $\underbrace{\text{very small}}_{\text{as } \eta\gg1}$ $t_f \to \eta\,\bar\nabla^2\bar b = -(\bar B_0\cdot\bar\nabla)\bar u$

$$\frac{\tau_{diff}}{\tau_u} \sim \frac{|\bar b|}{|\eta\bar\nabla^2\bar b|}\cdot\frac{U}{L} \sim \frac{L^{\cancel{2}}}{\eta}\cdot\frac{U}{\cancel{L}} = Re_m$$

Magn. f. adjusts to $\bar u$ instant.

$$\boxed{\eta\,\bar\nabla^2\bar b = -(\bar B_0\cdot\bar\nabla)\bar u}$$ induct. eq-n under the quasi-static approx.

$$|\bar b| \sim \frac{B_0 U}{\cancel{L}}\cdot\frac{L^{\cancel{2}}}{\eta} = Re_m B_0$$

$$\Rightarrow |\bar b| \ll B_0$$

In the quasi-static approx. the induced magn. f. is negligible compared to the applied one

$$\frac{1}{\rho}\,\bar{j}\times\bar{B} = \frac{1}{\rho}\,\bar{j}\times\bar{B}_0 = \frac{\sigma}{\rho}\left(\bar{E}+\bar{u}\times\bar{B}_0\right)\times\bar{B}_0 \longrightarrow \text{Lor. f.}$$

Helmholtz's th.:

$$\bar{E} = \underbrace{-\bar{\nabla}\varphi}_{\text{irrot.}\ \to\ \bar{\nabla}\times\ldots=0} + \underbrace{\bar{\nabla}\times\bar{a}}_{\text{solenoidal}\ \to\ \bar{\nabla}\cdot\ldots=0}$$

Use Faraday's law $\bar{\nabla}\times\bar{E} = -\frac{\partial\bar{b}}{\partial t}$,

$$\bar{\nabla}\times\bar{E} = \bar{\nabla}\times(\bar{\nabla}\times\bar{a}) \sim \frac{|\bar{b}|}{\tau} \sim \frac{Re_m B_0 U}{L}$$

$$\Rightarrow \bar{\nabla}\times\bar{a} \sim Re_m B_0 U$$

$$\bar{E} \approx -\bar{\nabla}\varphi$$

Lor. force: $\frac{\sigma}{\rho}\left(-\bar{\nabla}\varphi+\bar{u}\times\bar{B}_0\right)\times\bar{B}_0$

φ: electrostatic potential

$$\begin{cases} \frac{\partial\bar{u}}{\partial t}+(\bar{u}\cdot\bar{\nabla})\bar{u} = \frac{1}{\rho}\bar{\nabla}p+\nu\bar{\nabla}^2\bar{u}+\frac{\sigma}{\rho}\left(-\bar{\nabla}\varphi+\bar{u}\times\bar{B}_0\right)\times\bar{B}_0 \\ \text{Diverg. of the Ohm's law:} \\ \quad \bar{\nabla}\cdot\bar{j}=0=\sigma\left(-\bar{\nabla}^2\varphi+\bar{\nabla}\cdot(\bar{u}\times B_0)\right) \\ \quad \Rightarrow \bar{\nabla}^2\varphi = B_0\cdot(\bar{\nabla}\times\bar{u}) = \bar{B}_0\cdot\bar{\omega}, \\ \bar{\nabla}\cdot\bar{u}=0 \end{cases}$$

the quasi-static MHD eq-s in potentia formulation